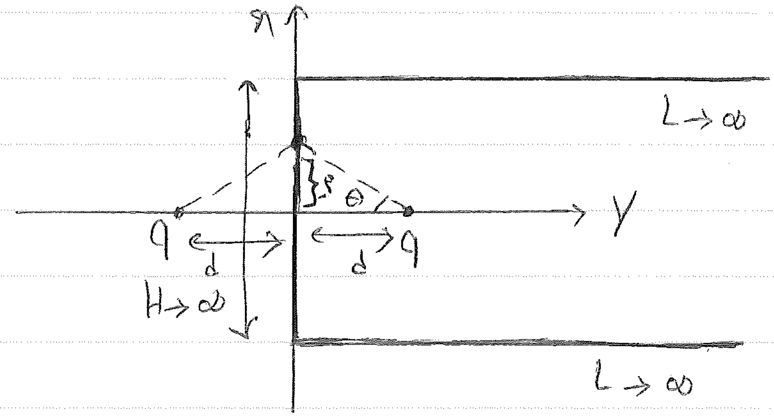
Problem Session 7 03/23/2018

(1) Use of Maxwell stress tensor to calculate the force between two equal point charges.

(2) Use of Maxwell stress tensor to calculate the force on a hemisphere of (i) a uniformly charged solid <u>insulating</u> sphere; (ii) a uniformly charged <u>perfectly conducting</u> sphere.

(1) A suitable region that contains one of the charges is as follows:




The force is given by the surface integral of the momentum tensor as $\vec{P}_{EM} = 0$ (Since $\vec{B} = 0$).

The only non-vanishing contribution to the surface integral is that on the xz plane. Since normal to this plane is $-\hat{y}$, we then have:

$$F_i = -\int_{xz\,plane} T_{i2}\, da \quad , \quad T_{i2} = \epsilon_0 \left(E_i E_2 - \tfrac{1}{2}\delta_{i2} E^2\right)$$

On the xz plane $da = \rho\, d\rho\, d\phi$, and:

$$E_1 = \frac{2q\rho}{4\pi\epsilon_0 (q^2+\rho^2)^{3/2}} \cos\phi \quad , \quad E_3 = \frac{2q\rho}{4\pi\epsilon_0 (q^2+\rho^2)^{3/2}} \sin\phi \quad , \quad E_2 = 0$$

The only non-zero $T_{i2}$ is $T_{22}$, where:

$$T_{22} = \frac{1}{2}\epsilon_0 E^2 = \frac{1}{2}\epsilon_0 \frac{q^2\rho^2}{4\pi^2\epsilon_0^2 (q^2+\rho^2)^3}$$

Therefore:

$$F_2 = \frac{-q^2}{8\pi^2\epsilon_0}\int_0^{2\pi} d\phi \int_0^{\infty} \frac{\rho^2}{(d^2+\rho^2)^3}\,\rho\, d\rho = \frac{-q^2}{4\pi\epsilon_0}\int_0^{\infty}\frac{\rho^3\, d\rho}{(d^2+\rho^2)^3} \underset{t^2 = \rho^2+d^2}{=} \frac{-q^2}{4\pi\epsilon_0}$$

$$\int_d^{\infty} \frac{t^2-d^2}{t^6}\, t\, dt = \frac{-q^2}{4\pi\epsilon_0}\left[\frac{-1}{2t^2} + \frac{d^2}{4t^4}\right]_d^{\infty} = \frac{q^2}{16\pi\epsilon_0 d^2}$$

This results in (as expected):

$$\boxed{\vec{F} = \frac{q^2}{4\pi\epsilon_0 (2d)^2}\,\hat{y}}$$

(2) (i) Insulating sphere. In this case, the charge is uniformly distributed throughout the sphere resulting in $\rho = \frac{Q}{\frac{4}{3}\pi R^3}$ ($R$ being the radius). Using Gauss's law, we find:

$$\begin{cases} \vec{E}(\vec{r}) = \dfrac{Q}{4\pi\epsilon_0 r^2}\,\hat{r} & r > R \\ \vec{E}(\vec{r}) = \dfrac{Qr}{4\pi\epsilon_0 R^3}\,\hat{r} & r \le R \end{cases}$$

A suitable closed surface that contains the upper hemisphere is the $xy$ plane and a hemispherical surface of radius $\infty$. The only non-zero contribution to the integral of $T_{ij}$ on this surface is that over the $xy$ plane. We note that the normal to this plane is $-\hat{z}$. Again, since $\vec{B} = 0$, we have $\vec{P}_{EM} = 0$ and the surface integral gives us the force on the upper hemisphere. Thus:

$$F_i = \int_{xy\ \text{plane}} -T_{i3}\, da$$

The only non-zero $T_{i3}$ is $T_{33}$, where:

$T_{33} = -\frac{1}{2}\varepsilon_0 E^2$ (since $E_3 = 0$ on the $xy$ plane)

This results in:

$$F_3 = \frac{1}{2}\varepsilon_0 \int_0^{2\pi} d\phi \int_0^{\infty} E^2 \rho\, d\rho = \frac{Q^2}{16\pi\varepsilon_0}\left[\int_0^R \frac{\rho^3 d\rho}{R^6} + \int_R^{\infty} \frac{\rho\, d\rho}{\rho^4}\right] \Rightarrow$$

$$\boxed{\vec{F} = \frac{3Q^2}{64\pi\varepsilon_0 R^2}\hat{z}}$$

(ii) Conducting sphere. In this case, the charge is distributed uniformly on the surface of the sphere. Hence:

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r} \quad r \geq R \quad , \quad \vec{E} = 0 \quad r < R$$

Repeating the same steps, we find:

$$\boxed{\vec{F} = \frac{Q^2}{32\pi\varepsilon_0 R^2}\hat{z}}$$